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Some calculations of gravity wave resistance in an inviscid stratified fluid

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A disk moves edgewise in a vertical plane with a sufficiently small constant velocity in a uniformly stratified inviscid fluid under gravity. The resultant hydrodynamic thrust on the disk is estimated. Configurations of some associated phase surfaces are shown. A transient Green's function is presented for the case when the Boussinesq approximation is not made, and also for an analogous case of inertial waves in a rotating fluid.

1. Introduction

A thin body (disk), symmetric with respect to the vertical y, z plane, moves within a stratified fluid on a steady path at an angle of inclination ϵ_0 to the vertical. The density of the fluid decreases exponentially upwards, and so in the presence of a gravity field wave motions of the fluid are excited. The results presented here represent a generalization of some earlier work by one of us (Warren 1960), for the case of strictly vertical motion. The theory is a particular case of the general treatment given by Lighthill (e.g. 1967).

For low speeds the results are relevant to meteorological problems, for example (see Richards 1962). The linearized equation which governs the modified pressure perturbation, p, for a dry inviscid ideal isothermal atmosphere is

$$[\partial^2/\partial t^2\{\nabla^2 - \gamma^2 g^2/4c^4 - c^{-2}\,\partial^2/\partial t^2\} + (\gamma-1)\,(g^2/c^2)\,\{\partial^2/\partial x^2 + \partial^2/\partial y^2\}]\,p = 0. \quad (1.1)$$

This equation is readily deduced from the results of Eckart (1960, chapter 4). The z axis is positive upwards. The modified pressure perturbation is related to the true pressure perturbation p_1 through the equation $p = (p_1/\bar{\rho})\exp(\gamma g z/2c^2)$, where c is the velocity of sound and $\bar{\rho}$ is a standard density. γ is the ratio of specific heats of the gas. For axes (x, y, z) which move with the body, that is, with a velocity $\mathbf{U} = (0, V, W)$, $\partial/\partial t$ is replaced by $\partial/\partial t - \mathbf{U}.\nabla$, which for steady motion reduces to $-\mathbf{U}.\nabla$. Then if $|\mathbf{U}| \ll c$ the resulting equation of steady motion is approximately

$$[(\mathbf{U}.\nabla)^2\{\nabla^2 - \beta^2/4\} + N^2\{\partial^2/\partial x^2 + \partial^2/\partial y^2\}]\,p = 0, \quad (1.2)$$

where $\beta = \gamma g/c^2$ and $N = (\gamma-1)^{\frac{1}{2}} g/c$ is the Väisälä–Brunt frequency. Relative to axes fixed in the disk, the equation of its hull is

$$x = \pm a\xi(y, z), \quad (1.3)$$

where $2a$ is the maximum thickness of a disk whose length and breadth are A and B respectively. Then $a \ll A$, $a \ll B$ and a small thickness parameter α is defined by $\alpha = 2a/A$. For sufficiently small α, linearization is valid and the boundary condition at the hull, which is a free slip condition, is approximately

$$\partial p/\partial x|_{x=\pm 0} = \pm a \exp(-\beta z/2)(\mathbf{U}.\nabla)^2 \xi(y,z). \tag{1.4}$$

The case for which $A\beta \ll 1$ $(\gamma g A/c^2 \ll 1)$ and $gA/|\mathrm{U}c| = O(1)$ is considered. That is, the stratification density gradient is small and the gravitational field is strong, and the Boussinesq approximation is made. Further remarks on this approximation will be made in § 4. The problem is then to search for a solution of the equation

$$[(\mathbf{U}.\nabla)^2 \nabla^2 + (\mathbf{N} \times \nabla)^2] p = 0, \tag{1.5}$$

[where $\mathbf{N} = N\hat{\mathbf{z}}$, ($\hat{\mathbf{z}}$ = unit vector upwards)], subject to

$$\partial p/\partial x|_{x=\pm 0} = \pm a(\mathbf{U}.\nabla)^2 \xi. \tag{1.6}$$

As is well known, this system does not possess a unique solution, since certain wave systems may be superimposed arbitrarily without destroying the validity of (1.5) or (1.6): the boundary condition at infinite distance is not well defined. To overcome this difficulty, Lighthill's (1967) method may be used, or, equivalently, the limit of the initial value problem may be taken. Thus equation (1.5) is replaced by

$$[(\partial/\partial t - \mathbf{U}.\nabla)^2 \nabla^2 + (\mathbf{N} \times \nabla)^2] p = 0. \tag{1.7}$$

Equations (1.6) and (1.7) are also appropriate for the case of an incompressible stratified fluid, where the density $\rho(z)$ decreases exponentially upwards; that is, $\rho(z) = \bar{\rho}\exp(-\beta z)$, and the Boussinesq approximation (equivalent to the limiting process $\beta \to 0, g \to \infty$, with $\beta g = N^2$ = constant) is again made.

2. The hydrodynamic forces on the disk

The boundary condition at large distances for the initial value problem is

$$\lim_{|\mathbf{r}|\to\infty} p(\mathbf{r},t) = \lim_{|\mathbf{r}|\to\infty} |\nabla p(\mathbf{r},t)| = 0 \tag{2.1}$$

for all finite t. Introduce the Fourier transforms

$$p(\boldsymbol{\kappa},t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, p(\mathbf{r},t)\, e^{-i\boldsymbol{\kappa}.\mathbf{r}}$$

and
$$\xi(l,m) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \xi(y,z)\, e^{-i(ly+mz)},$$

where
$$\boldsymbol{\kappa} = (\kappa, l, m) \quad \text{and} \quad \kappa = |\boldsymbol{\kappa}|, \quad \hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa}/\kappa.$$

Then equation (1.7) becomes

$$[(\partial/\partial t - i\mathbf{U}.\boldsymbol{\kappa})^2 \boldsymbol{\kappa}^2 + (\mathbf{N}\times\boldsymbol{\kappa})^2] p(\boldsymbol{\kappa},t) = 2a(\mathbf{U}.\kappa)^2\{(\mathbf{U}.\boldsymbol{\kappa})^2 - \mathbf{N}^2\}\xi(l,m), \tag{2.2}$$

when the boundary conditions (1.6) and (2.1) are taken into account. The motion may be supposed initiated at a time $t = 0$ when both $p(\kappa, 0)$ and $\partial p(\kappa, 0)/\partial t$ vanish. (This supposition is discussed further in § 4.) Then

$$p(\boldsymbol{\kappa}, t) = 2a(\mathbf{U}.\boldsymbol{\kappa})^2 (\mathrm{N}^2 - (\mathbf{U}.\boldsymbol{\kappa})^2)\, \xi(l, m).P(\boldsymbol{\kappa}, t)/\boldsymbol{\kappa}^2\sigma_1\sigma_2, \tag{2.3}$$

where

$$\sigma_{1,2} = \sigma_{1,2}(\boldsymbol{\kappa}) = \mathbf{U}.\boldsymbol{\kappa} \pm |\mathbf{N} \times \hat{\boldsymbol{\kappa}}| \tag{2.4}$$

and

$$P(\boldsymbol{\kappa}, t) = 1 + (\sigma_2 e^{i\sigma_1 t} - \sigma_1 e^{i\sigma_2 t})/(\sigma_1 - \sigma_2). \tag{2.5}$$

Let y', z' be co-ordinates in the y, z plane fixed in the disk and respectively parallel and normal to its direction of motion (figure 1). Let $D(t)$, $L(t)$ be the

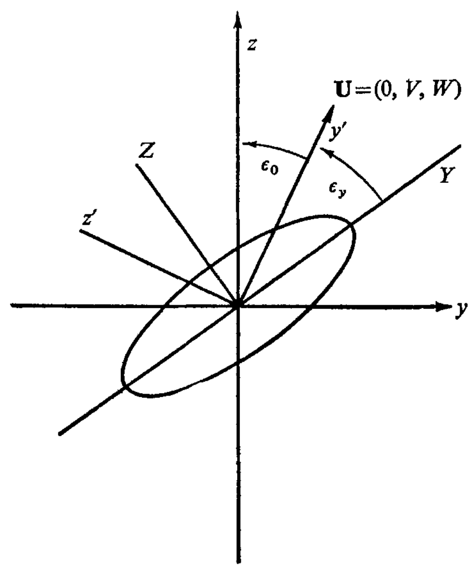

FIGURE 1. Co-ordinate axes and reference angles.

forces conjugate to the y' and z' directions respectively. In these co-ordinates the equation of the hull is $x = \xi_0(y', z')$ and so approximately

$$(D, L) = 2a \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz'\, p_1(0, y', z')\, (\partial\xi_0(y', z')/\partial y', \quad \partial\xi_0(y', z')/\partial z').$$

The force conjugate to the x direction is zero, by symmetry. This symmetry is essential to the analysis (and a great deal of algebraic and computational simplification results from this assumption). Hence approximately

$$\begin{aligned}(D, L) &= (a\bar{\rho}/4\pi^3) \int_V d\kappa' p(\boldsymbol{\kappa}', t) \int dy' \int dz' (\partial\xi_0/\partial y', \partial\xi_0/\partial z')\, e^{i(l'y' + m'z')} \\ &= (a\bar{\rho}/4i\pi^3) \int_V d\boldsymbol{\kappa}' p(\boldsymbol{\kappa}, t)\, \xi_0(-l', -m')\, (l', m'),\end{aligned}$$

and hence, by equation (2.3),

$$(D, L) = (a^2\rho/2i\pi^3) \int_V d\boldsymbol{\kappa} (\mathbf{U}.\boldsymbol{\kappa})^2 (\mathrm{N}^2 - (\mathbf{U}.\boldsymbol{\kappa})^2\, |\xi_0(l, m)|^2 P(\boldsymbol{\kappa}, t)\, (\boldsymbol{\kappa}^2\sigma_1\sigma_2)^{-1} (l, m), \tag{2.6}$$

when the prime is dropped. V denotes the $\boldsymbol{\kappa}$ wave-number space. This last integral expression may be simplified if spherical polar co-ordinates are used. Set

$$k = \kappa\cos\theta, \quad l = \kappa\sin\theta\sin\phi \quad \text{and} \quad m = \kappa\sin\theta\cos\phi.$$

Then the time dependent conjugate forces are given by

$$\begin{aligned}(D,L) = (a^2\bar{\rho}\mathrm{U}^2/2i\pi^3)\int_0^\pi d\theta\int_{-\pi}^{\pi} d\phi\int_0^\infty d\kappa\,\kappa^3\sin^4\theta\sin^2\phi \\ \times(N^2-\kappa^2U^2\sin^2\theta\sin^2\phi)\,|\xi_0|^2(\sin\phi,\cos\phi) \\ \times\{(1/\sigma_1\sigma_2)+(\sigma_2 e^{i\sigma_1 t}-\sigma_1 e^{i\sigma_2 t})/\sigma_1\sigma_2(\sigma_1-\sigma_2)\}.\end{aligned} \tag{2.7}$$

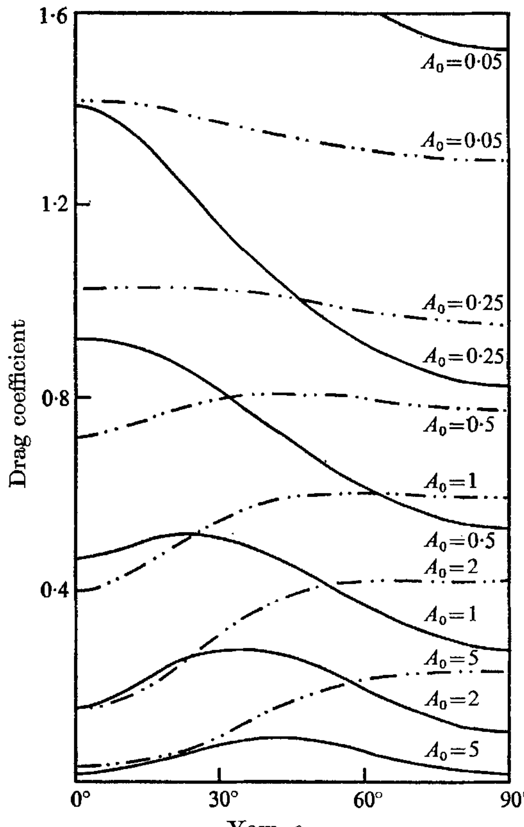

FIGURE 2. Force coefficient $D_0 = D/(\frac{1}{2}\bar{\rho}\pi^2a^2A_0^2b_0^2N^2)$ against yaw ϵ_y at various Froude numbers $A_0 = a_0N/U$ for model (i), equation (2.10). ($b_0/a_0 = \frac{1}{10}$.) ———, $\epsilon_0 = \frac{1}{2}\pi$; —·—, $\epsilon_0 = 0$.

Here $\qquad U = |\mathrm{U}|, \quad \xi_0 = \xi_0(\kappa\sin\theta\sin\phi, \kappa\sin\theta\cos\phi),$

and $\qquad \sigma_{1,2} = -U\kappa\sin\theta\sin\phi \pm N|1-\sin^2\theta\sin^2(\phi+\epsilon_0)|^{\frac{1}{2}},$

so that $\qquad \sigma_1\sigma_2 = U^2\kappa^2\sin^2\theta\sin^2\phi - N^2(1-\sin^2\theta\sin^2(\phi+\epsilon_0))$

and $\qquad \sigma_1-\sigma_2 = 2N|1-\sin^2\theta\sin^2(\phi+\epsilon_0)|^{\frac{1}{2}}.$

ϵ_0 is the angle which the path makes with the vertical: $\tan\epsilon_0 = V/W$. Now in the expression (2.7), the factor within the braces in the integrand is proper, but if the terms within these braces are considered separately the corresponding integrals are improper. However, equality is preserved if principal values are taken. In this event, the integral which derives from the term $(1/\sigma_1\sigma_2)$ vanishes (consider the ϕ-integral and set $\phi \to \phi+\pi$, and the result follows), while for a reasonably behaved function of κ, say $f(\kappa)$,

$$\lim_{t\to\infty}\int_0^\infty d\kappa f(\kappa)\, e^{i\sigma_1 t}/\sigma_1 = \pi i f(\kappa_s)\, H(\phi)/(U\sin\theta\,|\sin\phi|), \tag{2.8}$$

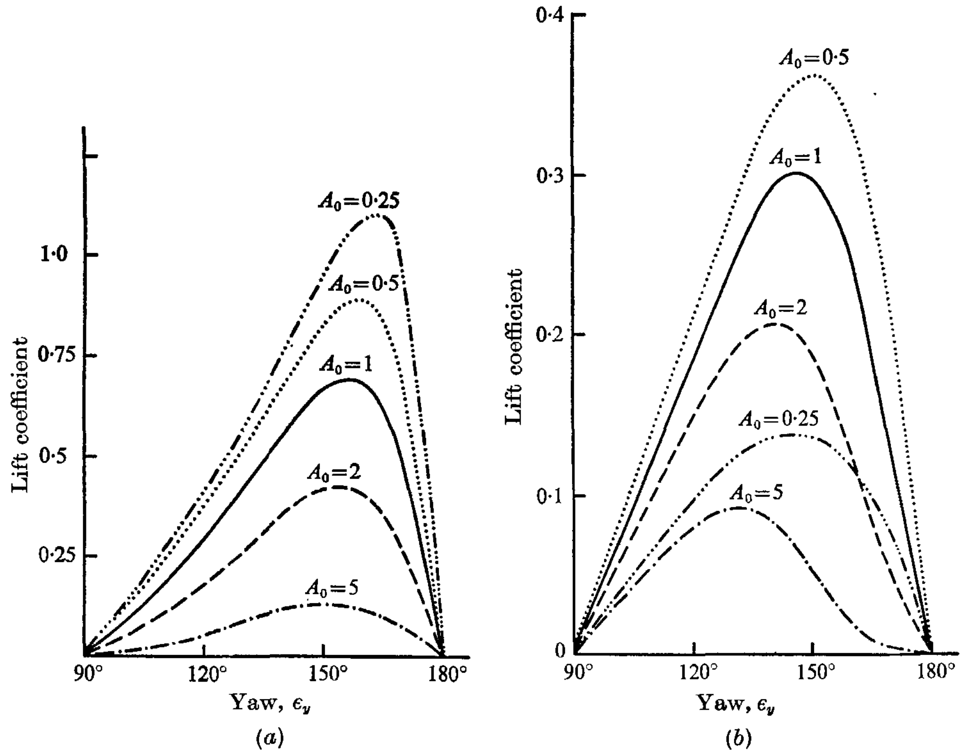

FIGURE 3. Force coefficient L_0 as for figure 2. (*a*) $\epsilon_0 = 90°$, (*b*) $\epsilon_0 = 0°$.

with a similar result for

$$\int_0^\infty d\kappa f(\kappa)\, e^{i\sigma_2 t}/\sigma_2.$$

$H(\phi)$ denotes the Heaviside unit function, and

$$\kappa_s = N|1-\sin^2\theta\sin^2(\phi+\epsilon_0)|^{\frac{1}{2}}/(U\sin\theta\,|\sin\phi|).$$

Hence if (D,L) now denotes the limiting form of the conjugate forces for large values of the time t, it follows that

$$(D,L) = (N^4\bar{\rho}a^2/4\pi^2U^2)\int_0^\pi d\theta\int_{-\pi}^{\pi} d\phi\,(1-\sin^2\theta\sin^2(\phi+\epsilon_0))$$
$$\times \sin^2\theta\,(\sin\phi,\cos\phi)\sin^2(\phi+\epsilon_0)\,|\xi_s|^2/(\sin^2\phi\,\mathrm{sgn}\,\phi), \tag{2.9}$$

where $\xi_s = \xi_0(\kappa_s\sin\theta\sin\phi,\, \kappa_s\sin\theta\cos\phi)$,

and $\mathrm{sgn}\,\phi = \phi/|\phi|.$

This formula is closely related to the wave resistance given by Warren for the axisymmetric case of vertical motion (Warren 1960, equation (19)). Put $\epsilon_0 = 0$ in (2.9) and change to Cartesian co-ordinates (l, m). Set

$$\xi(l, m) = \{(2\sin Bl/Bl)\}\,\xi_0(m)$$

and consider the limiting process $B \to 0$. Then the earlier result is recovered, except for a modification of the multiplying factor outside the force integral. This modification is not surprising since the boundary conditions for the thin

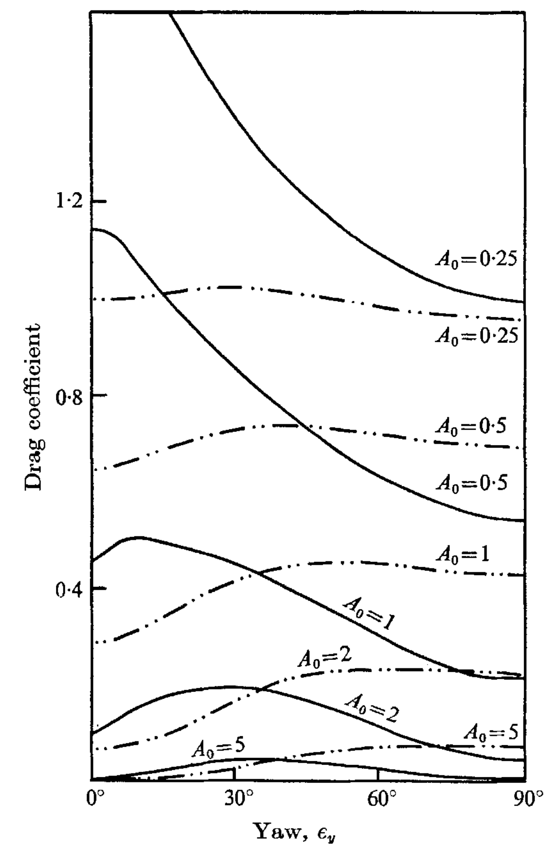

FIGURE 4. Force coefficient $D_0 = D/(\frac{1}{2}\bar{\rho}a^2A_0^2b_0^2N^2)$ against yaw ϵ_y at various Froude numbers $A_0 = a_0N/U$ for model (ii), equation (2.11). $(b_0/a_0 = \frac{1}{10}.)$ ———, $\epsilon_0 = 90°$; —·—, $\epsilon_0 = 0°$.

disk are different from those which are used in the axisymmetric case. It should be recalled that $\xi_0(y', z')$ refers to axes along and normal to the path. If $\xi_{00}(Y, Z)$ refers to axes of longitudinal and transverse symmetry of the disk respectively, and ϵ_y is the angle between the Y and y' axes (that is, ϵ_y is the yaw angle) then the relation between the Fourier transforms of ξ_0 and ξ_{00} is given by

$$\xi_0(l, m) = \xi_{00}(l\cos\beta_0 + m\sin\beta_0,\, m\cos\beta_0 - l\sin\beta_0).$$

For the purposes of calculation, disks whose contours are given by

$$\text{(i)}\quad \xi_{00}(Y, Z) = \exp\left(-Y^2a_0^{-2} - Z^2b_0^{-2}\right) \tag{2.10}$$

and

$$\text{(ii)}\quad \xi_{00}(Y, Z) = a_0^2b_0^2/(Y^2 + a_0^2)(Z^2 + b_0^2) \tag{2.11}$$

were selected. Units of length and time are defined by setting $U = 1$ and $n = 1$. Curves of the conjugate force coefficients are shown in figures 2 to 6 for various angles of ascent ϵ_0 and yaw angles ϵ_y. An interesting result arises for the case of horizontal motion, $\epsilon_0 = \frac{1}{2}\pi$. Here the force opposing motion is minimal when the disk moves broadside on, that is when the yaw ϵ_y is $\frac{1}{2}\pi$. Details of the numerical work involved in evaluating the formulae (2.9) are given in MacKinnon (1968).

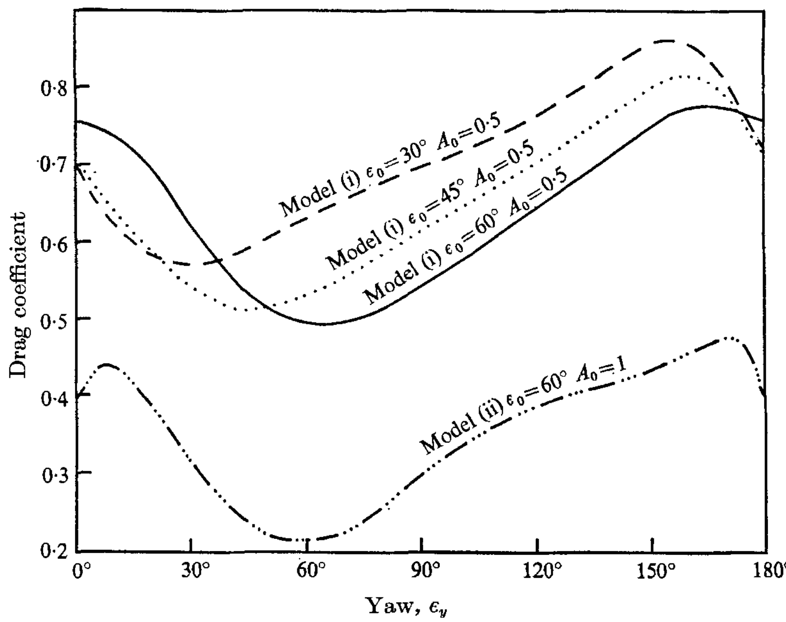

FIGURE 5. Some force coefficient curves for models (i) and (ii) for oblique ascent.

The computations were carried out on the London University Computer Atlas. These results are for bodies of infinite extent. The question has arisen of how they compare with the results for bodies of finite extent. No definite answer can be given here, but previous work (e.g. Warren 1960) suggests that provided the body presents no bluff contours to the current there will be no great change in the force coefficients.

3. The phase surfaces

Referring to (2.3), the modified pressure is given by

$$p(\mathbf{r},t) = (a/4\pi^3)\int_V d\kappa(\mathbf{U}.\boldsymbol{\kappa})^2(N^2-(\mathbf{U}.\boldsymbol{\kappa})^2)\,\xi_0(l,m)\,P(\boldsymbol{\kappa},t)\,e^{i\boldsymbol{\kappa}.\mathbf{r}}/\kappa^2\sigma_1\sigma_2. \quad (3.1)$$

As in the previous section, the integral is expressed as a triple integral in spherical polar co-ordinate form, and the integral $\int_0^\infty d\kappa$ over scalar radial wave-numbers κ is considered first. The path of integration is deformed into the complex κ plane, and the separate components of the integrand are defined as principal value

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integrals, as in (2.7). Then, considering the limit $p(\mathbf{r})$ of $p(\mathbf{r},t)$ as $t \to \infty$, the integral (3.1) yields the result

$$p(\mathbf{r}) = (\tfrac{1}{2}\pi^2 i)\int_{\Sigma_1} dSQ(\boldsymbol{\kappa})\, e^{i\boldsymbol{\kappa}_s.\mathbf{r}} + O(|\mathbf{r}|^{-3}), \tag{3.2}$$

where $\boldsymbol{\kappa}_s = \boldsymbol{\kappa}_s(\theta,\phi) = (\kappa_s \cos\theta, \kappa_s \sin\theta \sin\phi, \kappa_s \sin\theta\cos\phi)$,

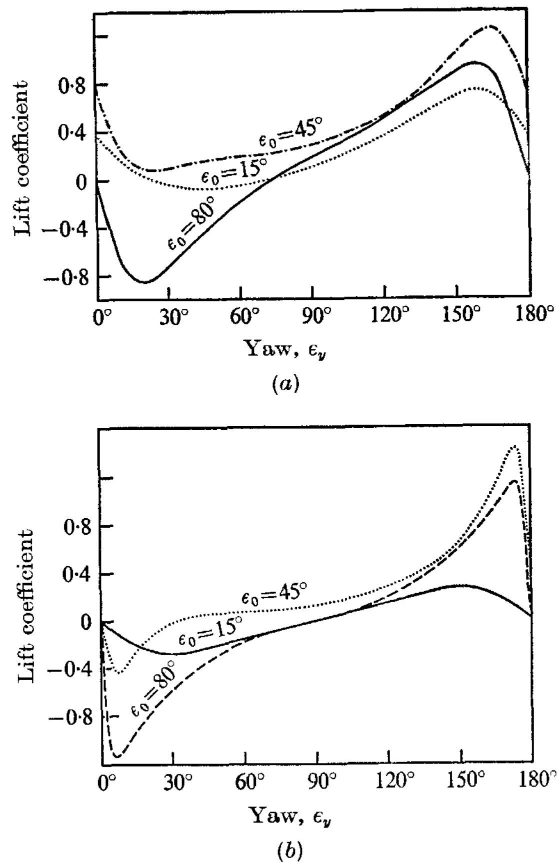

FIGURE 6. (a) Some coefficient curves for model (i) ($A_0 = 0{\cdot}5$). (b) Some lift coefficient curves for model (ii) ($A_0 = 1$).

and where $Q(\boldsymbol{\kappa})$ is regular on Σ_1. Σ_1 is that part of the surface of the unit sphere where

$$\operatorname{sgn}(\boldsymbol{\kappa}_s.\mathbf{r}) + \operatorname{sgn}(\boldsymbol{\kappa}_s.\mathbf{U}) = 0. \tag{3.3}$$

$\boldsymbol{\kappa}_s$ is given by the intersection of the cone

$$|\mathbf{N}\times\hat{\boldsymbol{\kappa}}| = c = \text{constant}$$

and the planes

$$\mathbf{U}.\boldsymbol{\kappa} = \pm c,$$

since $\boldsymbol{\kappa}_s$ satisfies $\sigma_{1,2}(\boldsymbol{\kappa}_s) = 0$. In terms of the polar and azimuth angles, for large $|\mathbf{r}|$,

$$p(\mathbf{r}) = \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi Q(\theta,\phi).\,|\operatorname{sgn}(\boldsymbol{\kappa}_s.\mathbf{r}) - \operatorname{sgn}(\boldsymbol{\kappa}_s.\mathbf{U})|\exp(i\boldsymbol{\kappa}_s(\theta,\phi).\mathbf{r}), \tag{3.4}$$

approximately. For a given point (θ, ϕ) on Σ_1, there is one and only one value of $\kappa_s\,(>0)$, and hence only one value of $\boldsymbol{\kappa}_s$, for which $\sigma_1\sigma_2$ vanishes. For large $|\mathbf{r}|$, the method of stationary phase yields a term $O(|\mathbf{r}|^{-1})$. Following Lighthill (1964, 1967), the phase surfaces are given by the condition that $\boldsymbol{\kappa}_s(\theta,\phi)\,.\,\mathbf{r}$ is stationary, and it follows that these surfaces are given by

$$\mathbf{r}_{\text{phase}}(\theta,\phi) = C(\partial\boldsymbol{\kappa}_s/\partial\theta \times \partial\boldsymbol{\kappa}_s/\partial\phi)/[\partial\boldsymbol{\kappa}_s/\partial\theta, \partial\boldsymbol{\kappa}_s/\partial\phi, \boldsymbol{\kappa}_s], \tag{3.5}$$

subject to the condition (3.3) where C is constant for a given phase surface. Calculations for some of these surfaces have been carried out and models have

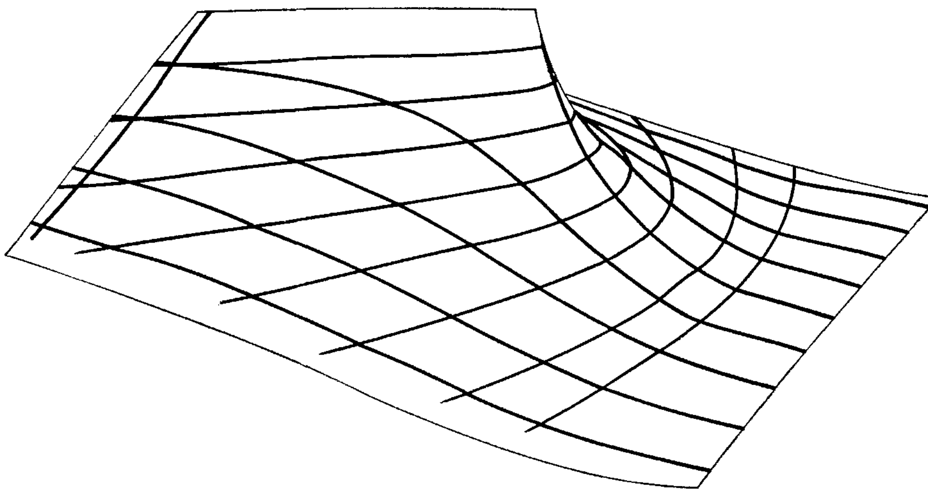

FIGURE 7. A phase surface for horizontal motion. (Lower sheet.)

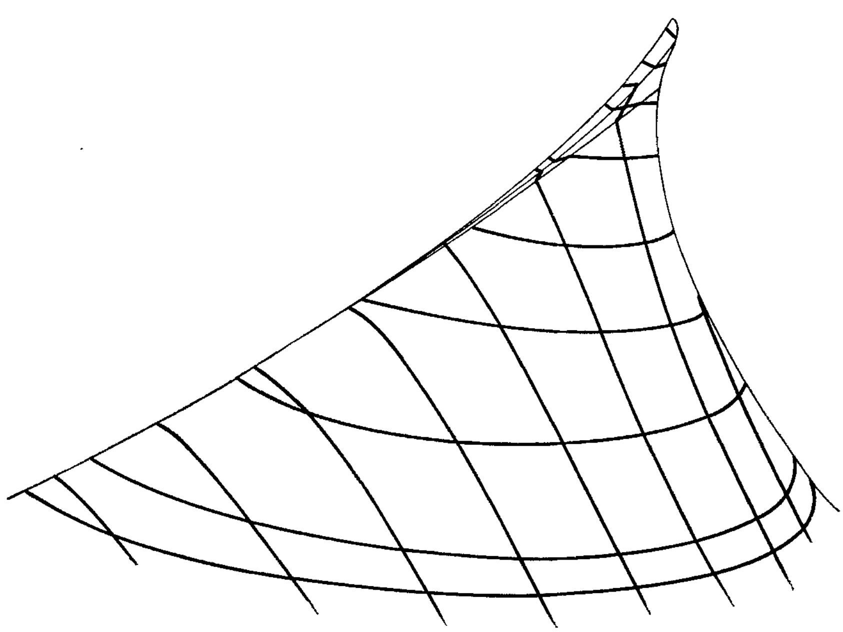

FIGURE 8. A phase surface for motion along a line inclined at 60° to the vertical.

been constructed for the cases of horizontal motion and for an ascent angle of $\epsilon_0 = 60^\circ$ (figures 7 and 8). Forward propagation of waves was found in this latter case, and in fact for all cases except for purely vertical ($\epsilon_0 = 0$) or horizontal motion ($\epsilon_0 = \frac{1}{2}\pi$). For details of the working the reader is referred to Mulley (1968). The vertical section made by the y, z plane through these surfaces agree with results for the two-dimensional case given by Rarity (1967), except that no forward propagation of waves was found when the disk moves horizontally. A further check is given by comparison with one of Lighthill's (1967) results for the case of vertical motion, $\epsilon_0 = 0$.

Finally, it is noted that if in (3.1) the substitutions $U = (0, 0, W)$, $x = r\cos\theta$, $y = r\sin\theta$ and $\xi_0(l, m) = \delta(l)\,\xi_0(m)$ are made, then an integration with respect to θ yields an expression for the pressure for the axisymmetric case with vertical motion. If the identification

$$J_0(kr) = \frac{1}{\pi}\int_0^\pi d\theta\, e^{ikr\cos\theta}$$

is made, this expression is seen to correspond with the form of the stream function obtained by Warren (1960).

4. Comment on the Boussinesq approximation for an incompressible fluid

If the Boussinesq approximation is not made it is well known that certain difficulties in the linearization of the basic equations can arise. For example, in the case of ascent, the wave amplitude of the fluid particle displacement increases exponentially upwards like $\exp(\frac{1}{2}\beta z)$, while in descent, the pressure and density perturbations increase like $\exp(\frac{1}{2}\beta z)$. Further difficulties arise in connexion with the initial value problem. In the case of vertical descent, Warren (1960) attempted to show that certain 'transients' may not decay, but ultimately may become large because of contributions from terms

$$O(t^{-m}\, e^{\beta_1 t}), \tag{4.1}$$

which occur in the stream function. However, this result is misleading because further work has shown that for strictly vertical motion m is infinite. Similarly for horizontal motion β_1 is zero. Nevertheless, for oblique ascent, the problem remains, and in this case the analysis below shows that m is finite and β_1 is positive. A similar result has been obtained by Fraenkel (1962, unpublished) who has shown that for vertical ascent within an infinite vertical duct or channel of finite width, the stream function contains a term $O(t^{-\frac{1}{2}}\, e^{\beta_1 t})$ where $\beta_1 > 0$. This suggests that oblique reflexion from the vertical walls of the duct plays an important part in the ultimate behaviour of transients. Consider the transient axisymmetric problem of a vertical line source immersed in an infinite incompressible stratified fluid, and make linear approximations in the usual way, but do not make the Boussinesq approximation. Choose cylindrical polar co-ordinate axes whose polar axis coincides with the z axis and put $(x^2+y^2)^{\frac{1}{2}} = r$. Let the strength of the source along the z axis be $f(z)\,g(t)$ where $f(z)$ vanishes outside a certain interval of the z axis containing the origin, and $g(t)$ is non-zero only for $0 < t < T$ say. Then the modified pressure $p(r, z, t)$ satisfies

$$\partial^2/\partial t^2(\partial^2/\partial r^2 + r^{-1}\partial/\partial r - \beta^2/4 + \partial^2/\partial z^2)\,p + N^2(\partial^2/\partial r^2 + r^{-1}\partial/\partial r)\,p = 0, \quad (4.2)$$

and the transformed modified pressure,

$$p(\lambda, m, \omega) = \int_0^\infty dt \int_{-\infty}^\infty dm \int_0^\infty dr\, r J_0(\lambda r)\, e^{-i(mz-\omega t)},$$

where $\operatorname{Im}\omega > 0$, is equal to

$$\frac{i\omega(N^2-\omega^2) f(\lambda)\, g(\omega)}{\omega^2(\lambda^2+m^2+\frac{1}{4}\beta^2) - N^2\lambda^2}, \quad (4.3)$$

where $f(\lambda)\, g(\omega)$ represents the transformed source strength. Inverting, the true pressure perturbation is given by

$$p_1(r, z, t) = \bar{\rho}\frac{\partial}{\partial t}\int_{-\infty}^\infty d\zeta f(\zeta) \int_0^t d\tau g(\tau)\, G(r, z-\zeta, t-\tau),$$

where

$$G(r, z, t) = \left(\frac{e^{-\frac{1}{2}\beta z}}{4\pi^2}\right)\int_{-\infty+ic}^{\infty+ic} d\omega \int_0^\infty d\lambda \int_{-\infty}^\infty dm \frac{(\omega^2-N^2)\,\lambda J_0(\lambda r)\, e^{i(mz-\omega t)}}{\omega^2(\lambda^2+m^2+\frac{1}{4}\beta^2) - N^2\lambda^2}, \quad (4.4)$$

where $c > 0$. Evaluate the m integral here by the calculus of residues. The λ integration may then be performed by using the identity

$$\int_0^\infty d\lambda (\lambda^2+b^2)^{-\frac{1}{2}}\lambda J_0(\lambda r) \exp\{-a(\lambda^2+b^2)^{\frac{1}{2}}\} = (r^2+a^2)^{-\frac{1}{2}}\exp\{-b(r^2+a^2)^{\frac{1}{2}}\},$$

where the real parts of a and b are positive (see, for example, Erdelyi *et al.* 1954, p. 9). Similar identities have been employed by Pierce (1963) and Row (1967) in studies of acoustic gravity waves in the atmosphere. The resulting expression for G is

$$(4\pi)^{-1}(r^2+z^2)^{-\frac{1}{2}} e^{-\frac{1}{2}\beta z}\int_{-\infty+ic}^{\infty+ic} d\omega\, e^{-i\omega t}(\omega^2-N^2)^{\frac{1}{2}}(\omega^2-N^2\cos^2\epsilon_0)^{-\frac{1}{2}}$$
$$\times \exp\{-\tfrac{1}{2}\beta(r^2+z^2)^{\frac{1}{2}}(\omega^2-N^2\cos^2\epsilon_0)^{\frac{1}{2}}(\omega^2-N^2)^{-\frac{1}{2}}\}, \quad (4.5)$$

where the roots have positive real parts and $\sin\epsilon_0 = r(r^2+z^2)^{-\frac{1}{2}}$. When either $z = 0$ or $r = 0$ it is seen that p_1 tends to zero at large distances for all values of the time. However, consider an observer who moves with a velocity $(U, 0, W)$, where neither U nor W are zero. Set $r = Ut$, $z = Wt$. Then

$$G(Ut, Wt, t) = \frac{1}{\pi}(U^2+W^2)^{-\frac{1}{2}} e^{-\frac{1}{2}\beta Wt} t^{-1}\int_{N\cos\epsilon_0}^{N} d\omega (N^2-\omega^2)^{\frac{1}{2}}(\omega^2-N^2\cos^2\epsilon_0)^{-\frac{1}{2}}$$
$$\times \sin\omega t\cos\{\tfrac{1}{2}\beta(U^2+W^2)^{\frac{1}{2}} t(\omega^2-N^2\cos^2\epsilon_0)^{\frac{1}{2}}(N^2-\omega^2)^{-\frac{1}{2}}\}$$
$$+\tfrac{1}{2}(U^2+W^2)^{-\frac{1}{2}} t^{-1}\exp[-\tfrac{1}{2}\beta t\{(U^2+W^2)^{\frac{1}{2}}+W\}]\,\delta(t), \quad (4.6)$$

where $\delta(t)$ is the Dirac delta function. (In expression (4.5), the residues at $\omega = N$ and $\omega = N\cos\epsilon_0$ vanish.) For large values of the time t, the integral in this expression for G is $O(t^{-\frac{1}{2}})$ or $O(t^{-1})$ according to whether

$$\omega - \tfrac{1}{2}\beta(U^2+W^2)^{-\frac{1}{2}}(\omega^2-N^2\cos^2\epsilon_0)^{\frac{1}{2}}(N^2-\omega^2)^{-\frac{1}{2}}$$

has or has not stationary points in the interval of integration. It follows that for large t, $p_1(Ut, Wt, t)$ is of the form (4.1) where m is positive and finite, and β_1 is negative if $W > 0$ (ascent) and positive if $W < 0$ (descent). Hence the dispersion of waves in an infinite uniformly stratified inviscid liquid is such that neither pressure waves which travel downwards nor fluid velocity perturbations which travel upwards are truly evanescent.

It is worth while to note that from (4.5) it follows that

$$p = (x^2+y^2+z^2)^{-\frac{1}{2}} J_0\{Ntz(x^2+y^2+z^2)^{-\frac{1}{2}}\} \tag{4.7}$$

is a solution of

$$\{\partial^2/\partial t^2 \nabla^2 + N^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)\} p = 0, \tag{4.8}$$

except at the origin. However, the behaviour of this solution at the origin is not a suitable (e.g. delta) function of the time. It is also worth while to note the analogous case of a uniform fluid which rotates as a rigid body about the z axis with angular velocity Ω. The pressure perturbation satisfies

$$[\partial^2/\partial t^2\{\partial/\partial r(r\,\partial/\partial r) + r\,\partial^2/\partial z^2\} + 4\Omega^2 r\,\partial^2/\partial z^2]\, p = 0 \tag{4.9}$$

(see e.g. Greenspan 1968, p. 22). Here of course no question of the Boussinesq approximation arises. The analysis similar to the one above shows that for a transient line source on the z axis the solution is given by

$$p = \bar{\rho} \int_{-\infty}^{\infty} d\zeta f(\zeta) \int_0^{\infty} d\tau\, g(\tau)\, G_\Omega(r, z-\zeta, t-\tau), \tag{4.10}$$

where

$$G_\Omega(r,z,t) = \frac{\partial^2}{\partial t^2}[(r^2+z^2)^{-\frac{1}{2}} J_0\{2\Omega tr(r^2+z^2)^{-\frac{1}{2}}\} H(t)]. \tag{4.11}$$

The phase surfaces of the solutions (4.7) and (4.11) are a series of right conical surfaces whose axes coincide with the z axis, their common apex being at the origin. These cones open outwards and away from the axis as time increases, in the case of gravity waves, and they collapse onto the axis in the case of inertial waves.

Finally, the method carries through for the general case of a rotating compressible isothermal fluid under gravity, when the equation for p is obtained by adding

$$4\Omega^2[\partial^2/\partial z^2 - \beta^2/4 - c^{-2}\,\partial^2/\partial t^2]\, p$$

to the left-hand side of (1.1), see Eckart (1960, pp. 95–7) and Greenspan (1968, p. 13). The Green's function is given by the expression (4.5) if in the integrand β and $\cos\epsilon_0$ are replaced by $(\beta^2 - 4c^{-2}\omega^2)^{\frac{1}{2}}$ and $(\cos^2\epsilon_0 + 4\Omega^2 N^{-2}\sin^2\epsilon_0)^{\frac{1}{2}}$ respectively, and shows that non-evanescent transients are again present. When gravity is absent ($N = \beta = 0$) the solution is given by (4.11) if within the square brackets t is replaced by $|t^2 - R^2c^{-2}|^{\frac{1}{2}}\,\mathrm{sgn}\,(ct - R)$ (where $R = (x^2+y^2+z^2)^{\frac{1}{2}} = (r^2+z^2)^{\frac{1}{2}}$), while for the incompressible case, $c = \infty$, the function corresponding to (4.7) is obtained by replacing N by $(N^2 + 4\Omega^2 r^2 z^{-2})^{\frac{1}{2}}$, showing that the conical phase surfaces open outwards if $N > 2\Omega$, and collapse if $N < 2\Omega$.

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REFERENCES

ECKART, C. 1960 *Hydrodynamics of Oceans and Atmospheres*. Oxford: Pergamon.

ERDELYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F. G. 1954 *Tables of Integral Transform*, vol. II. New York: McGraw-Hill.

FRAENKEL, L. E. 1962 Unpublished notes on waves in stratified fluids.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

LIGHTHILL, M. J. 1964 *J. Inst. Maths. Applic.* **1**, 1.

LIGHTHILL, M. J. 1967 *J. Fluid Mech.* **27**, 725.

MACKINNON, R. F. 1968 Ph.D. Thesis, London University.

MULLEY, R. 1968 D.I.C. Dissertation, Imperial College, London.

PIERCE, A. D. 1963 *J. Acoustic. Soc. Am.* **35**, 1798.

RARITY, B. S. H. 1967 *J. Fluid Mech.* **30**, 329.

RICHARDS, J. M. 1962 *J. Fluid Mech.* **11**, 369.

ROW, R. V. 1967 *J. Geophys. Res.* **72**, 1599.

WARREN, F. W. G. 1960 *J. Fluid Mech.* **7**, 209.